Non periodic orbits in a four dimensional symplectic map

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Abstract

We numerically investigate the projections of non-periodic orbits in a 4-D symplectic map composed of two coupled 2-D maps. We investigate how the structures produced on the projection plane, change as the coupling parameter grows. This change is given numerically by an empirical law.

1 Introduction

We study the structure of non periodic orbits in a 4-D symplectic map composed of two coupled 2-D maps. The 4-D map is

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= x_2 - \nu\sin(x_1+x_2) - \mu[1-\cos(x_1+x_2+x_3+x_4)] \\ x_3' &= x_3 + x_4 \\ x_4' &= x_4 - \kappa\sin(x_3+x_4) - \mu[1-\cos(x_1+x_2+x_3+x_4)] \end{aligned} \pmod{2\pi} \qquad (1)$$

The periodic orbits of this map have been studied by Contopoulos and Giorgilli [1].

The projections of non periodic orbits on the plane (x_1, x_2) have a structure which has been studied by Skokos [2] (Paper I). In Paper I we saw that the projections of the successive consequents of the non periodic orbit with initial conditions $x_1 = 3, x_2 = x_3 = x_4 = 0$ for $\nu = 10^{-3}, \kappa = 10^{-1}$ and $\mu = 10^{-5}$ are ordered in a particular way. The points form n curves with successive consequents on every m curves, which correspond to the resonance $\frac{m}{n}$ that appears when $x_2 \approx -2\pi\frac{m}{n} \pmod{2\pi}$. This happens because in the limiting case $\mu = 0, \nu = 0$ for $x_2 = -2\pi\frac{m}{n} \pmod{2\pi}$ and $x_1 \in [-\pi, \pi)$ we get on the plane (x_1, x_2) n points and any two successive points differ by m. So the non periodic orbit is influenced by all possible resonances $\frac{m}{n}$ on the plane (x_1, x_2) as seen in FIG.1.

Now we study how this structure changes as μ gets bigger, which means that the system goes away from the two uncoupled 2-D maps.

2 The width of the zones of the resonances on the plane $(\mathbf{x_1}, \mathbf{x_2})$

As we see in FIG.1 the zones where the curves of the resonances are formed do not have the same width. For example the width along the x_2-axis of the zone of resonance $\frac{3}{14}$ is larger than the one of $\frac{4}{19}$ or $\frac{5}{23}$. We empirically measure the width of some resonances by defining their limits at the points where the corresponding curves become more or less parallel to the

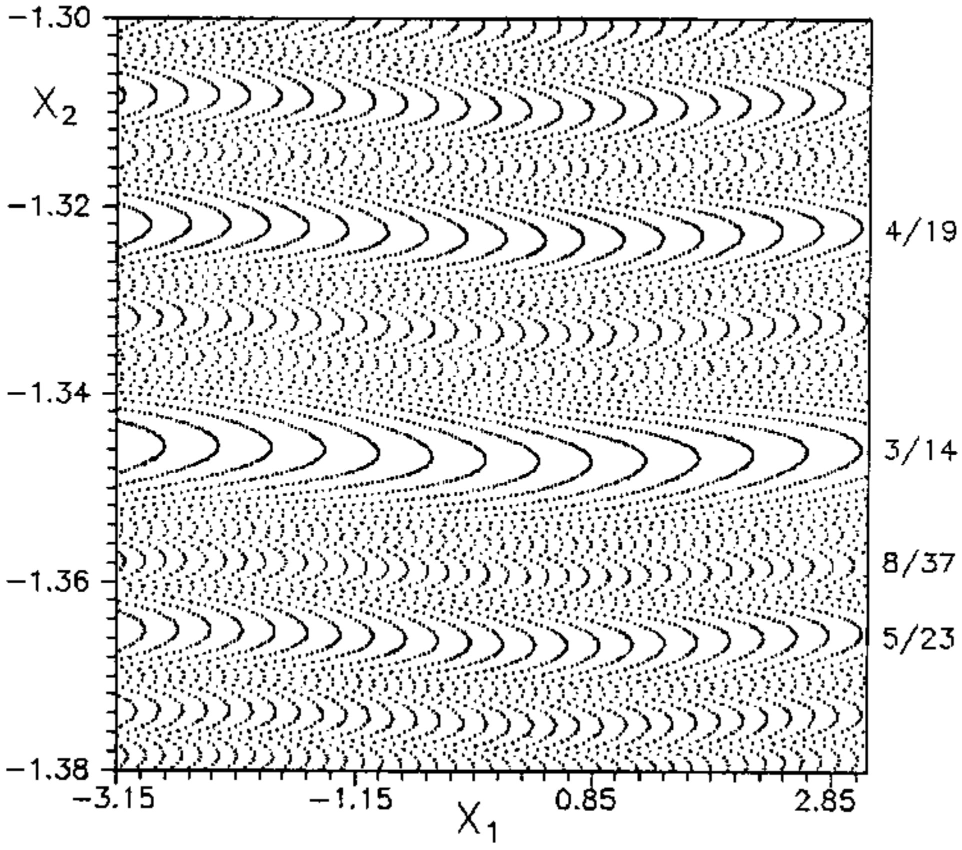

Figure 1: Successive consequents of the orbit with initial conditions $x_1 = 3$, $x_2 = x_3 = x_4 = 0$ for $\nu = 10^{-3}, \kappa = 10^{-1}$ and $\mu = 10^{-5}$ in a region containing the resonances $\frac{m}{n} = \frac{4}{19}, \frac{3}{14}, \frac{8}{37}$ and $\frac{5}{23}$.

x_1–axis. Resonances of high order, which means large denominator, have small width and we cannot define its limits easily. On the other hand the resonance $\frac{1}{1}$, which corresponds to $x_2 \approx 0$, has a large width, but its zone is deformed because of the main island around $x_1 = x_2 = 0$.

So for $\nu = 10^{-3}, \kappa = 10^{-1}$ and $\mu = 10^{-5}$ we measure the widths of zones of resonances with denominator starting from 2 up to 20.

In FIG.2 we plot the width (W) of the zone of the resonances $\frac{m}{n}$ as a function of $\frac{m}{n}$. Some kind of structure is evident and an approximate symmetry with respect to the line $\frac{m}{n} = \frac{1}{2}$ is seen. Connecting the points of resonances with the same denominator we see that their width is almost constant, thus we conclude that the width of the zone of a resonance depends mainly on its denominator and it gets smaller as the denominator grows. In FIG.3 we plot the width (W) as a function of the denominator. Also we plot a best fitting line of the form

$$W = A\frac{1}{n^B} \tag{2}$$

where we found

$$A = 0.190896 \quad , \quad B = 1.04929 \tag{3}$$

Since $B \approx 1$ we could say that the width of the zone of a resonance is almost inversely proportional to its denominator. This law explains the structure we see in FIG.2 : the points of resonances lay roughly on some lines. For instance the resonances $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$, $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \dots$, $\frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \dots$, and so on are along certain lines. In general the points of the sequence of resonances :

$$\frac{m}{n}, \frac{m+\alpha}{n+\beta}, \frac{m+2\alpha}{n+2\beta}, \dots \tag{4}$$

where $m, n, \beta \in N^*, \alpha \in N, m < n$ and $0 \leq \alpha \leq \beta$, form in FIG.2 a line

$$W = K(\frac{m}{n}) + L \tag{5}$$

where $K, L \in R$. By assuming that $B = 1$ in Eq. (2) we can define K and L in formula (5). So we get

$$K = \frac{\beta}{m\beta - n\alpha} A \tag{6}$$

and

$$L = -\frac{\alpha}{m\beta - n\alpha} A \tag{7}$$

In FIG.2 some lines of the form (5) for the appropriate values of K and L derived from (6) and (7), are plotted. We see that the lines are in good agreement with the empirically found points.

Since the sum $\sum_1^\infty \frac{1}{n}$ is infinite, while the total interval of x_2 is finite (2π), we can see resonances up to some upper value of the denominator. Empirically we find that this value $n_{0,emp}$ is

$$n_{0,emp} \approx 75 \tag{8}$$

Since the zone of the resonance $\frac{m}{n}$ is formed when $x_2 \approx -2\pi\frac{m}{n} \pmod{2\pi}$ and its width is given from (2) for $B = 1$ we consider that its zone is the interval

$$x_2 \in [-2\pi\frac{m}{n} - \frac{W}{2}, -2\pi\frac{m}{n} + \frac{W}{2}] \pmod{2\pi} \tag{9}$$

In order to find how many resonances up to a certain denominator are needed to cover the segment $x_2 \in [-\pi, \pi)$, we make a grid of the values of x_2 with step 0.0001 and we find the highest denominator n_0 we need in order to put all the values of the grid, at least in one segment of the form (9). We find that

$$n_0 \approx 65 \tag{10}$$

We see that the values of $n_{0,emp}$ and n_0 are close enough. Thus although we found that the width of the zone of a resonance is inversely proportional to its denominator, using

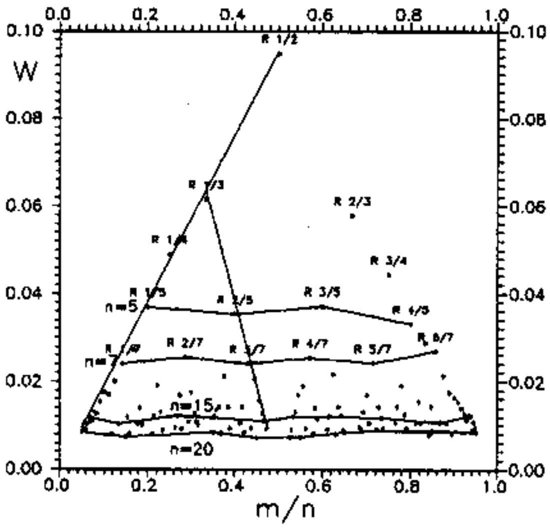

Figure 1: The width W of the zones of resonances with denominators from 2 up to 20, as a function of $\frac{m}{n}$. The widths of some resonances are marked. The points of resonances with denominators $5, 7, 15$ and 20 are line connected. Some lines of the form (5), corresponding to sequences of resonances of the form $\frac{m}{n}, \frac{m+\alpha}{n+\beta}, \frac{m+2\alpha}{n+2\beta}, \ldots$ for $m = 1, n = 1, \alpha = 0, \beta = 1$ and $m = 1, n = 3, \alpha = 1, \beta = 2$ are plotted

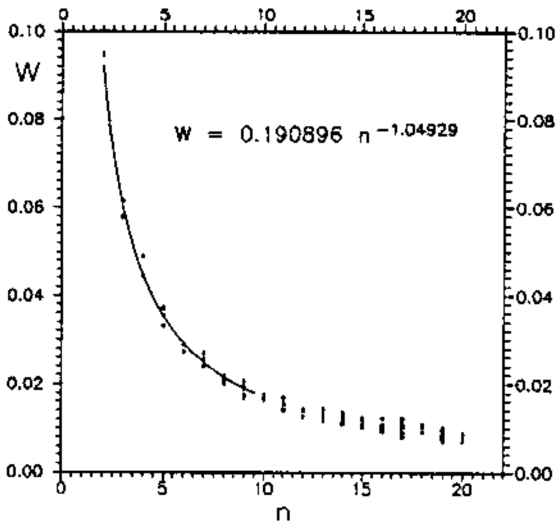

Figure 2: The width W of the zones of the resonances $\frac{m}{n}$ as a function of the denominator n. For every n we have a number of points which correspond to the various values of the numerator m. The best fitting line $W = 0.190896\, n^{-1.04929}$ is also plotted

resonances with denominator up to 20, this is valid for all the resonances in the plane (x_1, x_2).

If we increase the coupling parameter μ, we find the same structure as the one described in Paper I, although the number of the iterations of the orbit with initial conditions $x_1 = 3, x_2 = x_3 = x_4 = 0$, needed to cover the plane (x_1, x_2) decreases. For large values of μ $(\mu > 10^{-1})$ this structure is no longer valid because there is only a small number of points on the plane (x_1, x_2) and the zones of the resonances are not clearly formed.

For $\nu = 10^{-3}, \kappa = 10^{-1}$ and $\mu = 5 \times 10^{-5}$ by doing the same analysis as above we find that the law (2) is still valid with

$$A = 0.311524 \quad , \quad B = 1.0735 \tag{11}$$

and

$$n_{0,emp} \approx 46 \quad , \quad n_0 \approx 40 \tag{12}$$

For $\nu = 10^{-3}, \kappa = 10^{-1}$ and $\mu = 10^{-3}$ we get

$$A = 0.678511 \quad , \quad B = 0.980165 \tag{13}$$

and

$$n_{0,emp} \approx 18 \ , \ n_0 \approx 18 \tag{14}$$

Thus the value of B is always almost 1. On the other hand we find that the value of A increases approximately as a power of μ :

$$A = 4.53275\,\mu^{0.273411} \tag{15}$$

So the resonance widths are given by the approximate formula

$$W(\mu, \frac{m}{n}) \approx 4.53\,\frac{\mu^{0.27}}{n} \tag{16}$$

3 Conclusions

We studied how the structure of non periodic orbits, of a 4–D map composed of two coupled 2–D maps, changes on the plane (x_1, x_2), by varying the coupling parameter.

We saw that the projections on the plane (x_1, x_2) of the points of non periodic orbits are influenced strongly by the resonances of the corresponding 2–D map. Thus parts of the orbit are in zones parallel to the x_1–axis where $x_2 \approx -2\pi\frac{m}{n}$ (mod 2π). The n curves are formed with the successive consequents on every m curves. The width of these zones is given by the empirical law (16). The width of the zone of a certain resonance is inversely proportional to its denominator and it increases with the coupling parameter μ.

References

[1] Contopoulos G. and Giorgilli A., (1988) *Meccanica*, **23** , **19**

[2] Skokos Ch., (1993) *Proc. First Panhellenic Astron. Meeting , Athens* , p. **491**, *(in Greek)*